

# A MATHEMATICAL CONSIDERAION OF VORTEX THINNING IN 2D TURBULENCE

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**ABSTRACT.** In two dimensional turbulence, vortex thinning process is one of the attractive mechanism to explain inverse energy cascade in terms of vortex dynamics. By direct numerical simulation to the two-dimensional Navier-Stokes equations with small-scale forcing and large-scale damping, Xiao-Wan-Chen-Eyink (2009) found an evidence that inverse energy cascade may proceed with the vortex thinning mechanism. The aim of this paper is to analyze the vortex-thinning mechanism mathematically (using the incompressible Euler equations), and give a mathematical evidence that large-scale vorticity gains energy from small-scale vorticity due to the vortex-thinning process.

## 1. INTRODUCTION

“Vortex thinning” is one of the most important mechanism for two-dimensional turbulence. In [21], Xiao-Wan-Chen-Eyink investigated inverse energy cascade in steady-state two-dimensional turbulence by direct numerical simulation of the two-dimensional Navier-Stokes equations with small scale forcing and large scale damping. The two dimensional Navier-Stokes equations are described as follows:

$$(1.1) \quad \begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u &= -\nabla p + f, & t \geq 0, x \in [-\pi, \pi]^2 \\ \operatorname{div} u &= 0 \\ u(0) &= u_0 \end{aligned}$$

where  $u = u(t, x) = (u_1(x, t), u_2(x, t))$ ,  $p = p(x, t)$  and  $f = f_1(x_1, x_2), f_2(x_1, x_2)$  denote the velocity field, the pressure function of the fluid and the external force respectively. The case  $\nu = 0$ , we call the Euler equations. In their numerical work, they used an alternative equations (adding a damping term), and found strong evidence that inverse energy cascade may proceed with vortex thinning mechanism. According to their evidence, there is a tensile turbulent stress in directions parallel to the isolines of small-scale vorticity. In their analysis, the following large-scale strain tensor was introduced:

$$\bar{S}_\ell := \frac{1}{2} [(\nabla \bar{u}_\ell) + (\nabla \bar{u}_\ell)^T],$$

where  $\bar{u}_\ell$  is large-scale velocity defined by

$$\bar{u}_\ell(x, t) := \int_{\mathbb{R}^2} G_\ell(r) u(x + r, t) dr$$

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with  $G_\ell(r) = \ell^{-2}G(r/\ell)$  and  $G(r) = \sqrt{6/\pi}\exp(-6|r|^2)$ . From  $\bar{S}_\ell$ , we can define a quantity of “deformation work”, the rate of work being done locally by the large-scale strain as it acts against the small-scale stress. Thus the small-scale circular vortex will be stretched into elliptical shape. According to Kelvin’s theorem of conservation of circulation, the magnitude of the velocity around the vortex decrease, and then its energy should be reduced. The energy lost by the small-scale vortex is transferred to the large scale. In fact, the Reynolds stress created by the thinned vortex is primarily along the stretching direction, positive. Hence, the deviatoric stress is positively aligned with the large-scale strain (this energy transfer mechanism can be justified by (3.4) and (3.11) in [21]). It does negative work, namely, negative eddy viscosity. For more detail of their argument, see [21, Section 3 and Section 5].

The aim of this paper is to analyze such vortex-thinning mechanism mathematically (without “eddy viscosity”), and give a mathematical evidence that large-scale vorticity gains energy from small-scale vorticity due to the vortex-thinning process. In our consideration, for the sake of simplicity, we neglect the viscosity and the forcing terms. Thus we work with the inviscid vorticity equations (equivalently, the Euler equations) in the whole space:

$$(1.2) \quad \begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0, & t \geq 0, x \in \mathbb{R}^2 \\ \omega(0) &= \omega_0 \quad \text{with} \quad \int_{\mathbb{R}^2} \omega_0(x) dx = 0, \end{aligned}$$

where  $u = \nabla^\perp \Delta^{-1} \omega$  and the vorticity  $\omega$  is defined as  $\omega = \partial_1 u_2 - \partial_2 u_1$ . Throughout this paper we always handle smooth initial vorticity with compact support, thus the corresponding smooth solution always exists globally in time. Since the mean-value of the initial vorticity is zero, we see that the initial energy  $\|u(0)\|_{L^2}$  is always finite. Note that the first rigorous results for the existence theorem to (1.2) were proved in the framework of Hölder spaces by Gyunter [9], Lichtenstein [15] and Wolibner [20]. More refined results using a similar functional setting were obtained subsequently by Kato [10, 11], Swann [19], Bardos-Frisch [2], Ebin [7], Chemin [4], Constantin [5] and Majda-Bertozzi [16] among others. Here we just refer the Kato’s existence theorem. Let  $H^s$  ( $s \geq 0$ ) be inhomogeneous Sobolev spaces.

**Theorem 1.1.** ([11]) *Let  $s > 1$  and  $\omega_0 \in H^s(\mathbb{R}^2)$ . Then there is a unique solution  $\omega \in C([0, \infty); H^s(\mathbb{R}^2))$  of (1.2).*

We have to mention that the relation between the vortex-thinning process and palinstrophy ( $H^1$ -norm of the vorticity) has already been studied. In [1, Section 6.3] (see also [17]), they examined 2D vorticity equations with odd (in both  $x_1$  and  $x_2$ ) type of initial vorticities and measured these palinstrophy. From their argument we know that palinstrophy is one of the key to see 2D turbulence. On the other hand, some of mathematicians have showed that there is an initial vorticity in  $H^1$  such that the value of  $\|\nabla \omega(t)\|_{L^2}$  (palinstrophy) instantaneously blows up. More precisely, Bourgain-Li [3] and Elgindi-Jeong [8] constructed solutions to the 2D-Euler equations which exhibit norm inflation in  $H^1$  (see also [18]). We essentially use their construction of the initial vorticity. Let us now define “vortex thinning” mathematically. In order to give it, we need to define “Lagrangian flow  $\Phi$ ” as follows:

$$\partial_t \Phi(t, x) = u(t, \Phi(t, x)) \quad \text{with} \quad \Phi(0, x) = x \in \mathbb{R}^2$$

Note that the vortex-thinning is one of the “large deformation gradient” which is already described in [3, Proposition 3.4].

**Definition 1.2.** First we choose  $M > 1$  and fix it. We call “ $\Phi$  has a vortex thinning effect at a point  $x \in \text{supp } \omega_0(\Phi^{-1})$  and a time  $t$  with direction  $T$  (unit vector)” if and only if

$$\partial_T(\Phi(x, t) \cdot T) \geq M.$$

We now set a large-scale vorticity  $\omega_0^L$  and a small-scale vorticity  $\omega_0^S$  such that  $\omega_0^L, \omega_0^S \in C^\infty(\mathbb{R}^2)$ ,  $\text{supp } \omega_0^L \cap \text{supp } \omega_0^S = \emptyset$ ,  $|\omega_0^L|, |\omega_0^S| \lesssim 1$ , both of them are odd in  $x_1$  and  $x_2$  (we consider more general vorticity cases in the next section). More precisely, let us give the large-scale initial vorticity in the polar coordinate  $(r, \theta)$  as defined in [8, (5)] (see also [3, (3.4)] and [14]):

$$(1.3) \quad \omega_0^L(r, \theta) := \chi_N(r)\psi(\theta),$$

where  $\chi$  and  $\psi$  are smooth bump functions, namely,

$$\chi_N(r) := \begin{cases} 1 & \text{for } r \in [N^{-1}, N^{-1/2}] \\ 0 & \text{for } r \notin [N^{-1}/2, 2N^{-1/2}] \end{cases} \quad \text{and } \psi(\theta) := \begin{cases} 1 & \text{for } \theta \in [\pi/4, \pi/3] \\ \frac{1}{2} & \text{for } \theta \in [\pi/5, 9\pi/24] \\ 0 & \text{for } \theta \notin [\pi/6, 5\pi/12]. \end{cases}$$

We choose  $\omega_0^S$  such that  $\text{supp } \omega_0^S \subset B(N^{-1}) := \{x \in \mathbb{R}^2 : |x| < N^{-1}\}$  and

$$\{(h, h) : n^{-1} \leq h \leq n^{-1}(\log N)^{K\tau^*}\} \subset \text{supp } \omega_0^S$$

for some  $n > N$ . The meaning of the constants  $K$  and  $\tau^*$  are the same as in [8]. In this case  $H^1$ -norm (palinstorophy) may be large provided by large  $N$ . In [3] they figured out that this kind of odd vorticity creates large deformation gradient. But they employed a contradiction argument and thus we cannot figure out what kind of deformation and where it occurs (see Proof of Proposition 3.4 in their paper). Nevertheless, with the help of Elgindi-Jeong’s argument in [8], we can clarify it. The main result is as follows:

**Theorem 1.3.** *For any  $M > 1$ , there is  $N_0 > 0$  and  $\tau^* > 0$  such that if  $N > N_0$ , then there is  $t \in (0, \tau^* \log \log N / \log N]$ , at least either of the following two cases must occurs.*

- *Small-scale vortex-thinning:*

$$\partial_{e_2}(\Phi(x, t) \cdot e_2) \geq M^{1/2}$$

for some  $x \in \text{supp } \omega_0^S$ , where  $e_2 = (0, 1)$ .

- *Large-scale vortex-thinning (but it stretches only tail part):*

$$\partial_T(\Phi(x, t) \cdot T) \geq M^{1/2}$$

for some  $x \in \text{supp } \omega_0^L \cap D$  and  $T = \frac{1}{\sqrt{2}}(1, 1)$ , where

$$D = B\left(\alpha\tau^*N^{-1/2}\frac{\log \log N}{\log N}\right)$$

with sufficiently large positive constant  $\alpha > 0$ . (In this case we can rephrase  $\omega_0^L \cdot \chi_D$  as the small-scale vorticity, while  $\omega_0^L \cdot (1 - \chi_D)$  as the large-scale vorticity parts.)

*Proof.* The idea is to use Elgindi-Jeong's argument [8] with minor modifications. By (10) in [8], any line segment

$$\{(r, \theta_0) : N^{-1} \leq r \leq N^{-1/2}\}$$

evolve in a way that it intersects each circle  $\{r = r_0\}$  for  $N^{-5/6} \leq r_0 \leq N^{-4/6}$ . Recall that

$$I(t, r_0) := \{0 \leq \theta \leq \pi/2 : (r_0, \theta) \in R(t)\}$$

with

$$R(t) := \Phi(t, V) \cap A,$$

$$A := \{r : N^{-5/6} \leq r \leq N^{-4/6}\} \quad \text{and} \quad V := \{(r, \theta) : \omega_0^L(r, \theta) \geq 1/2\}.$$

We now consider two cases as in [8]:

- The case I: there exists a time moment  $t_0$  such that for more than half of  $r_0 \in [N^{-5/6}, N^{-4/6}]$  in the Haar measure, we have  $|I(t_0, r_0)| \leq M^{-1}$ .
- The case II: for all time  $t_0 \in [0, \tau^* \log \log N / \log N]$ , at least half of  $r_0 \in [N^{-5/6}, N^{-4/6}]$  in the Haar measure, we have  $|I(t_0, r_0)| \geq M^{-1}$ .

Let us now first consider the case I. In this case, diagonal direction of vortex thinning effect to  $\omega^L$  itself occurs. Let

$$\partial \text{cone}^+ := \{(r, \theta) : r > 0, \theta = 9\pi/24\}, \quad \partial \text{cone}^- := \{(r, \theta) : r > 0, \theta = \pi/5\}$$

and  $\partial \text{cone} = \partial \text{cone}^+ \cup \partial \text{cone}^-$ . Let  $n = (1/\sqrt{2})(-1, 1)$ . In this case there exists a time moment  $t_0$  and there are two points  $x_1 \in \partial \text{cone}^+ \cap \text{supp } \omega_0^L$  and  $x_2 \in \partial \text{cone}^- \cap \text{supp } \omega_0^L$  such that

$$|(\Phi(x_1, t_0) - \Phi(x_2, t_0)) \cdot n| \approx |\Phi(x_1, t_0) - \Phi(x_2, t_0)| \lesssim M^{-1} N^{-m}$$

with

$$\Phi(x_1, t_0) \cap \partial B(N^{-m}) \neq \emptyset \quad \text{and} \quad \Phi(x_2, t_0) \cap \partial B(N^{-m}) \neq \emptyset$$

for some  $m \geq 9/12$  (this  $9/12$  comes from “more than half of  $[N^{-5/6}, N^{-4/6}]$  in the Haar measure”). If both  $x_1$  and  $x_2$  satisfy

$$x_1, x_2 \in \partial \text{cone} \cap \left( B(N^{-1/2}) \setminus B(N^{-m} M^{-1/2}) \right),$$

then we can estimate the distance between  $x_1$  and  $x_2$  as

$$|(x_1 - x_2) \cdot n| \gtrsim N^{-m} M^{-1/2},$$

where  $n = (1/\sqrt{2})(-1, 1)$ . Thus by the mean value theorem, there is a point  $y \in \text{supp } \omega_0(\Phi^{-1})$  such that

$$|\partial_n(\Phi^{-1}(y, t_0) \cdot n)| \gtrsim M^{1/2}.$$

By the inverse function theorem with volume preserving, we have

$$(1.4) \quad |\partial_\tau(\Phi(x, t_0) \cdot \tau)| \gtrsim M^{1/2}$$

with  $x = \Phi^{-1}(y, t_0)$ . This is the desired estimate. If at most one point  $x_1$  satisfies

$$x_1 \in (B(N^{-1/2}) \setminus B(N^{-m} M^{-1/2})) \cap \partial \text{cone}$$

for some  $m > 9/12$ , then the other point must be

$$x_2 \in B(N^{-m} M^{-1/2}) \cap \partial \text{cone}.$$

In this case we choose a third point  $x_3$  to be

$$x_3 \in B(N^{-1}) \subset B(\epsilon N^{-m})$$

for sufficiently small  $\epsilon > 0$  (we choose  $\epsilon$  so that  $\epsilon > N^{-1+m}(\log N)^{C\tau}$ ). Then by (10) in [8], we see

$$\Phi(t_0, x_3) \in B(N^{-1}(\log N)^{C\tau}) \subset B(\epsilon N^{-m})$$

for large  $N > 0$ . Note that the constants  $C$  and  $\tau$  have the same meaning in [8]. Thus we have

$$|(x_2 - x_3) \cdot \tau| \lesssim N^{-m}(M^{-1/2} - \epsilon)$$

and

$$|(\Phi(t_0, x_2) - \Phi(t_0, x_3)) \cdot \tau| \gtrsim N^{-m}(1 - \epsilon).$$

By the mean-value theorem, we finally have

$$(1.5) \quad |\partial_\tau(\Phi \cdot \tau)| \gtrsim M^{1/2}(1 - \epsilon).$$

This is the desired estimate.

*Remark 1.4.* At least,  $|I(t_0, r_0)| \leq M^{-1}$  never occur in

$$r_0 \in \left[ \alpha\tau^* N^{-1/2} \frac{\log \log N}{\log N}, N^{-1/2} \right]$$

for sufficiently large  $\alpha > 0$ . In fact, since  $\omega(t, x) = \omega_0(\Phi^{-1}(t, x))$ ,  $\text{supp } \omega_0 \subset B(N^{-1/2})$ ,  $|\omega_0(x)| \leq 1$ , we have the following a priori velocity estimate:

$$\begin{aligned} |u(t_0, x)| &= |\nabla^\perp \Delta^{-1} \omega(t_0, x)| \leq |\nabla^\perp \Delta^{-1} \omega_0(\eta^{-1}(t_0, x))| \\ &\lesssim \sup_{\substack{\tilde{\Phi} \in (C^\infty(\mathbb{R}^2))^2 \\ \det D\tilde{\Phi} = 1}} \int_{[0, \infty)^2} \omega_0(\tilde{\Phi}(y)) \frac{dy}{|y - x|} \lesssim \int_{B(N^{-1/2})} \frac{1}{|x|} dx \lesssim N^{-1/2}. \end{aligned}$$

Let

$$\begin{aligned} x_1 &\in \partial \text{cone}^+ \cap \left( B(N^{-1/2}) \setminus B(\alpha\tau^* N^{-1/2} \log \log N / \log N) \right), \\ x_2 &\in \partial \text{cone}^- \cap \left( B(N^{-1/2}) \setminus B(\alpha\tau^* N^{-1/2} \log \log N / \log N) \right) \end{aligned}$$

with sufficiently large constant  $\alpha > 0$ . By the above a priori velocity estimate, we have

$$(1.6) \quad |\Phi(t_0, x_1) - \Phi(t_0, x_2)| \geq |x_1 - x_2| - 2N^{-1/2}t \gtrsim |x_1 - x_2|$$

for  $t_0 \in [0, \tau \log \log N / \log N)$ . Thus  $|I(t_0, r_0)| \geq M^{-1}$  provided by sufficiently large  $\alpha$ .

Next we consider the case II. In this case  $x_2$ -direction of vortex thinning effect to the small-scale vortex occurs, while large-scale vorticity  $\omega_0^L$  creates large scale strain. Let us choose two points  $x_1, x_2 \in \text{supp } \omega_0^S$  as (the constants  $K$  and  $\tau^*$  have the same meaning in [8])

$$x_1 = (n^{-1}, n^{-1}) \quad \text{and} \quad x_2 = ((\log N)^{K\tau^*} n^{-1}, (\log N)^{K\tau^*} n^{-1})$$

for some  $n > N$ . Now we recall Zlatos's velocity estimate [22] (just extend it to the whole space case):

**Theorem 1.5.** *Let  $\omega(t, \cdot)$  be odd in  $x_1$  and  $x_2$ . Then for  $x \in [0, 1/2]^2$ , we have*

$$u^i(t, x)/x_i = (-1)^i Q(t, x) + B_i(t, x)$$

with

$$Q(t, x) = \frac{4}{\pi} \int_{[2x_1, \infty) \times [2x_2, \infty)} \frac{y_1 y_2}{|y|^2} \omega(t, y) dy$$

and  $|B_i| \leq C\|\omega\|_\infty(1 + \log(1 + x_{3-i}/x_i))$  for  $i = 1, 2$ .

By the same argument as in [8] (the constants  $c_M$  and  $C_M$  have the same meaning in there), we have

$$Q(t, x_2) \geq c_M \log N$$

and the  $B_i(t, x)$ -term can be neglected. Thus we have

$$\partial_t \Phi_2(t, x_2) \geq C_M \log N \Phi_2(t, x_2).$$

Let  $e_2 = (0, 1)$ . Then we see that

$$\begin{aligned} \frac{(\Phi(x_2) - \Phi(x_1)) \cdot e_2}{(x_2 - x_1) \cdot e_2} &\geq \frac{(\log N)^{C_M} (\log N)^{K\tau^*} - (\log N)^{C\tau}}{(\log N)^{K\tau^*} - 1} \\ &= \frac{(\log N)^{C_M} - (\log N)^{C\tau - K\tau^*}}{1 - (\log N)^{-K\tau^*}}. \end{aligned}$$

In this case we choose  $K$  such that  $C - K < 0$  and choose sufficiently large  $N$  so that  $(\log N)^{C_M} \geq M$ . By the mean-value theorem, we have

$$(1.7) \quad \partial_{e_2}(\Phi \cdot e_2) \approx (\log N)^{C_M} \gtrsim M.$$

□

## 2. GENERAL VORTICITY SETTING

In this section we extend the vortex-thinning mechanism to general vorticity cases. Since we need to require finite energy, mean-zero vorticity condition  $\int \omega_0 = 0$  should be required. Otherwise, the energy becomes infinite due to the slowly decaying velocity. Let  $\partial_t \Phi(x, t) = u(\Phi(x, t), t)$  be a solution to the Euler equations (vorticity equations) (1.2) with the initial vorticity  $\text{rot } u_0 = \omega_0^S + \omega_0^L$ , and let  $\partial_t \Psi(x, t) = (u + v)(\Psi(x, t), t)$  also be a solution to the Euler equations (1.2) with the initial vorticity  $\text{rot } u_0 + \text{rot } v_0$  (in this case  $v_0$  is a perturbation, and assume  $\omega_0^P := \text{rot } v_0$  has compact support). By [3, Lemma 4.1], we have

$$\begin{aligned} \sup_{0 < t < 1} |\partial_T(\Phi \cdot T) - \partial_T(\Psi \cdot T)| &\leq \sup_{0 < t < 1} |D\Phi(t, x) - D\Psi(t, x)| \\ &\leq \left( \sup_{0 < t < 1} \|v(t)\|_\infty + \sup_{0 < t < 1} \|\nabla v(t)\|_\infty \right) \exp \left\{ \sup_{0 < t < 1} \|\nabla u(t)\|_\infty \right\}. \end{aligned}$$

By the Sobolev embedding,

$$(2.1) \quad \|v(t)\|_\infty + \|\nabla v(t)\|_\infty \leq C_1(s) \|v(t)\|_{H^s} \quad \text{for } s > 2,$$

where  $C_1$  is a positive constant satisfying  $C_1(s) \rightarrow \infty$  as  $s \rightarrow 2$ . Moreover, by the continuity on initial velocity in  $H^s$  ( $s > 2$ ), we have

$$\sup_{0 < t < 1} \|v(t)\|_s \leq C_2(s, \|u_0\|_s, \|u_0 + v_0\|_s) \|v_0\|_s,$$

where  $C_2$  is a positive constant satisfying  $C_2(s, \|u_0\|_s, \|u_0 + v_0\|_s) \rightarrow \infty$  as  $s \rightarrow 2$ . Continuity of the solution map for the Euler equations in Sobolev spaces  $H^s$  for  $s > 2$  is of course well known (see e.g., Ebin-Marsden [6], Kato-Lai [12], Kato-Ponce [13] and also [18]). Thus if the perturbation  $v_0$  is controlled as

$$C_1 C_2 \|v_0\|_s \leq C_3 M^{1/2} \quad \text{for some } s > 2$$

with some constant  $C_3 > 0$  determined by (1.4), (1.5) and (1.7), then we get the same vortex thinning mechanism to the initial velocity  $u_0 + v_0$ . This means that a

distorted symmetry case (measured in  $H^s$ ) also keeps the vortex-thinning process in a short time interval.

Vorticity far from the origin (we call “remainder part”) does not strongly affect to the vortex-thinning process which is occurring near the origin. In this case we just apply “gluing the patches argument”, Lemma 5.2 in [3]. Let

$$(2.2) \quad \begin{aligned} \partial_t \tilde{\omega} + \tilde{u} \cdot \nabla \tilde{\omega} &= 0, & t \geq 0, x \in \mathbb{R}^2 \\ \tilde{\omega}(0) &= f \quad \text{with} \quad f = \omega_0^L + \omega_0^S + \omega_0^P, \\ \partial_t \tilde{\Phi} &= \tilde{u}(\tilde{\Phi}), \end{aligned}$$

where  $\tilde{u} = \nabla^\perp \Delta^{-1} \tilde{\omega}$  and

$$(2.3) \quad \begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0, & t \geq 0, x \in \mathbb{R}^2 \\ \omega(0) &= f + g \quad \text{with} \quad g = \omega_0^R, \\ \partial_t \Phi &= u(\Phi), \end{aligned}$$

where  $u = \nabla^\perp \Delta^{-1} \omega$ . Here we need to assume  $\omega_0^R$  is in  $L^p \cap L^1$  ( $p > 2$ ) with  $\|\omega_0^R\|_{L^p} + \|\omega_0^R\|_{L^1} \lesssim 1$ . In this case we have

$$\begin{aligned} |u(t, x)| &= |\nabla^\perp \Delta^{-1} \omega(t, x)| \leq |\nabla^\perp \Delta^{-1} (f + g)(\Phi^{-1}(t, x))| \\ &\leq \sup_{\substack{\tilde{\Psi} \in (C^\infty(\mathbb{R}^2))^2 \\ \det D\tilde{\Psi} = 1}} \int_{\mathbb{R}^2} (f + g)(\tilde{\Psi}(y)) \frac{dy}{|y - x|} \leq \beta \end{aligned}$$

for some positive constant  $\beta > 0$ . By the above a priori velocity estimate, it is reasonable to assume

$$d(\text{supp } f, \text{supp } g) \geq \beta.$$

In this case, the supports of  $\tilde{\omega}(t)$  and  $\omega^R(t)$  are always disjoint in  $t \in [0, 1]$ . But we moreover need to assume that

$$d(\text{supp } f, \text{supp } g) \geq R_\epsilon (> \beta),$$

where  $R_\epsilon$  is already defined in Lemma 5.2 in [3]. With a minor modification of the proof of Lemma 5.2 in [3] (see also Remark 3.2 in this paper), we immediately have the following:

**Theorem 2.1.** *Let  $s > 1$  be fixed. Also let  $D = \{x : d(x, \text{supp } f) < \beta\}$  and  $\omega_f := \chi_D \omega$ . For any sufficiently small  $\epsilon > 0$ , we have*

$$\sup_{0 \leq t \leq 1} \|\tilde{\omega}(t) - \omega_f(t)\|_{H^s} < \epsilon$$

*provided that  $R_\epsilon > 0$  is sufficiently large.*

By the above theorem with the Sobolev embedding (2.1), we can easily show that the initial vorticity  $f + g$  also create the vortex-thinning process.

### 3. ENERGY TRANSFER FROM SMALL SCALE VORTEX TO LARGE SCALE VORTEX

In this section we give an evidence of energy transfer from small-scale vortex to large-scale vortex. For the small-scale vortex  $\omega^S$ , we assume a simple vortex-thinning process:  $\omega^S(x, t) = \omega_0^S(Mtx_1, (Mt)^{-1}x_2)$  for the sake of simplicity. Let

$$u^L := \nabla^\perp \Delta^{-1} \omega^L \quad \text{and} \quad u^S := \nabla^\perp \Delta^{-1} \omega^S.$$

We can measure the energies of each scale vortices. In this section we also assume that the mean value of the each scale vortices are zero, thus the energies of the each

vortices are also finite. Also assume that  $\text{supp } \omega_0^S \cap \text{supp } \omega_0^L = \emptyset$ . In this case we just directly take the  $L^2$ -norm to  $u^S$  and  $u^L$ .

**Theorem 3.1.** *Let  $\omega^S + \omega^L$  be a solution to (1.2) with initial vorticity  $\omega_0^S + \omega_0^L$ . For the initial vorticity  $\omega_0^S + \omega_0^L$  or  $-\omega_0^S + \omega_0^L$ , then we have the following energy estimate:*

$$(\|u_0^L\|_{L^2}^2 + \|u_0^S\|_{L^2}^2)^{1/2} - \|u^S(t)\|_{L^2} \leq \|u^L(t)\|_{L^2}.$$

Moreover, for fixed  $t \in (0, 1]$ ,

$$\|u^S(t)\|_{L^2} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

These estimates are the evidence of the energy-transfer mechanism.

*Proof.* If the velocity interaction is negative, namely,

$$(3.1) \quad \int u_0^L \cdot u_0^S < 0,$$

then we replace  $\omega_0^S$  to  $-\omega_0^S$ . In this case, the above integration (3.1) becomes positive. By the enstrophy conservation, and the disjoint supports, we see

$$\|\omega^L(t)\|_{L^2} = \|\omega_0^L\|_{L^2} \quad \text{and} \quad \|\omega^S(t)\|_{L^2} = \|\omega_0^S\|_{L^2}.$$

By the mean-zero vortex  $\omega_0^S$ , there is  $\Omega_0 \in C_c^\infty$  such that  $\partial_1 \Omega_0 = \omega_0^S$ . By taking the Fourier transform, we have

$$\begin{aligned} \|u_0^S((Mt)^{-1} \cdot, Mt \cdot)\|_{L^2}^2 &\leq \int \frac{|\hat{\omega}_0^S(\xi_1, \xi_2)|^2}{((Mt\xi_1)^2 + ((Mt)^{-1}\xi_2)^2)^{1/2}} d\xi \\ &\leq \frac{1}{Mt} \int |\hat{\Omega}_0(\xi_1, \xi_2)|^2 d\xi \rightarrow 0 \quad \text{as } M \rightarrow \infty. \end{aligned}$$

Also by the energy conservation to the incompressible Euler flow, we see

$$\begin{aligned} \|u_0^L\|_{L^2}^2 + \|u_0^S\|_{L^2}^2 &\leq \|u_0^L\|_{L^2}^2 + 2 \int u_0^L \cdot u_0^S + \|u_0^S\|_{L^2}^2 \\ &= \|u_0^L + u_0^S\|_{L^2}^2 = \|u^L(t) + u^S(t)\|_{L^2}^2 \\ &\leq (\|u^L(t)\| + \|u^S(t)\|_{L^2})^2 \end{aligned}$$

By the above estimate, the large scale vorticity gains energy from the small-scale vorticity. □

*Remark 3.2.* By the same argument as in the previous section, even if we add  $g$  in (2.3) to the initial vorticity, we can still get the energy-transfer process provided by sufficiently small  $\epsilon > 0$  and large  $R_\epsilon$ . We use the same notations in Section 2. Let  $u_f = \nabla^\perp \Delta^{-1} \omega_f$ ,  $u_g = \nabla^\perp \Delta^{-1} \omega_g$  and  $\omega_g(t, x) = g(\Phi^{-1}(t, x))$ . Note that

$$\omega = \omega_f + \omega_g = f(\Phi^{-1}) + g(\Phi^{-1})$$

and  $u_f$  satisfies

$$\partial_t u_f = (u_f \cdot \nabla) u_f + (u_g \cdot \nabla) u_f = -\nabla p_f$$

with some scalar function  $p_f$ . Set  $\eta = u_f - \tilde{u}$ . Then we have

$$(3.2) \quad \partial_t \eta + (\eta \cdot \nabla) u_f + (\tilde{u} \cdot \nabla) \eta + (u_g \cdot \nabla) u_f = -\nabla p_\eta$$

with some scalar function  $p_\eta$ . If the supports of  $\omega_g$  and  $\omega_f$  are far from each other, then  $\|(u_g \cdot \nabla) u_f\|_{L^2}$  is small enough. By the usual well-posedness theorem with a commutator estimate, then we can control  $\|u_f\|_{H^s}$  for  $s = 3, 4, 5, \dots$  (we use the



initial vorticity  $f + g$ , and the norm in  $H^s$  is independent of the distance between  $f$  and  $g$ ). Thus we have

$$\|\nabla \omega_f\|_\infty \leq \|\omega_f\|_{H^2} \leq \|\omega_f + \omega_g\|_{H^2} \lesssim \|f + g\|_{H^2}$$

and also

$$\|\nabla u_f\|_\infty \leq \|\omega_f\|_{H^3} \leq \|\omega_f + \omega_g\|_{H^3} \lesssim \|f + g\|_{H^3}$$

for  $t \in [0, 1]$ . We multiply  $\eta$  to (3.2) on both sides, integrate on  $\mathbb{R}^2$  and with some algebra, we obtain the following energy inequality:

$$\partial_t \|\eta\|_2^2 \lesssim \|\nabla u_f\|_\infty \|\eta\|_2^2 + \|\eta\|_2 \|(u_g \cdot \nabla) u_f\|_{L^2}.$$

Thus the energy  $\|\eta\|_2$  is small if  $\|(u_g \cdot \nabla) u_f\|_{L^2}$  is sufficiently small.

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